

Total monochromatic connection of graphs*

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Abstract

A graph is said to be *total-colored* if all the edges and the vertices of the graph are colored. A path in a total-colored graph is a *total monochromatic path* if all the edges and internal vertices on the path have the same color. A total-coloring of a graph is a *total monochromatically-connecting coloring* (*TMC-coloring*, for short) if any two vertices of the graph are connected by a total monochromatic path of the graph. For a connected graph G , the *total monochromatic connection number*, denoted by $tmc(G)$, is defined as the maximum number of colors used in a TMC-coloring of G . These concepts are inspired by the concepts of monochromatic connection number $mc(G)$, monochromatic vertex connection number $mvc(G)$ and total rainbow connection number $trc(G)$ of a connected graph G . Let $l(T)$ denote the number of leaves of a tree T , and let $l(G) = \max\{l(T) \mid T \text{ is a spanning tree of } G\}$ for a connected graph G . In this paper, we show that there are many graphs G such that $tmc(G) = m - n + 2 + l(G)$, and moreover, we prove that for almost all graphs G , $tmc(G) = m - n + 2 + l(G)$ holds. Furthermore, we compare $tmc(G)$ with $mvc(G)$ and $mc(G)$, respectively, and obtain that there exist graphs G such that $tmc(G)$ is not less than $mvc(G)$ and vice versa, and that $tmc(G) = mc(G) + l(G)$ holds for almost all graphs. Finally, we prove that $tmc(G) \leq mc(G) + mvc(G)$, and the equality holds if and only if G is a complete graph.

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1 Introduction

In this paper, all graphs are simple, finite and undirected. We refer to the book [3] for undefined notation and terminology in graph theory. Throughout this paper, let n and m denote the order (number of vertices) and size (number of edges) of a graph, respectively. Moreover, a vertex of a connected graph is called a *leaf* if its degree is one; otherwise, it is an *internal vertex*. Let $l(T)$ and $q(T)$ denote the number of leaves and the number of internal vertices of a tree T , respectively, and let $l(G) = \max\{l(T) \mid T \text{ is a spanning tree of } G\}$ and $q(G) = \min\{q(T) \mid T \text{ is a spanning tree of } G\}$ for a connected graph G . Note that the sum of $l(G)$ and $q(G)$ is n for any connected graph G of order n . A path in an edge-colored graph is a *monochromatic path* if all the edges on the path have the same color. An edge-coloring of a connected graph is a *monochromatically-connecting coloring* (*MC-coloring*, for short) if any two vertices of the graph are connected by a monochromatic path of the graph. For a connected graph G , the *monochromatic connection number* of G , denoted by $mc(G)$, is defined as the maximum number of colors used in an MC-coloring of G . An *extremal MC-coloring* is an MC-coloring that uses $mc(G)$ colors. Note that $mc(G) = m$ if and only if G is a complete graph. The concept of $mc(G)$ was first introduced by Caro and Yuster [6] and has been well-studied recently. We refer the reader to [4, 8] for more details.

As a natural counterpart of the concept of monochromatic connection, Cai et al. [5] introduced the concept of monochromatic vertex connection. A path in a vertex-colored graph is a *vertex-monochromatic path* if its internal vertices have the same color. A vertex-coloring of a graph is a *monochromatically-vertex-connecting coloring* (*MVC-coloring*, for short) if any two vertices of the graph are connected by a vertex-monochromatic path of the graph. For a connected graph G , the *monochromatic vertex connection number*, denoted by $mvc(G)$, is defined as the maximum number of colors used in an MVC-coloring of G . An *extremal MVC-coloring* is an MVC-coloring that uses $mvc(G)$ colors. Note that $mvc(G) = n$ if and only if $diam(G) \leq 2$.

Actually, the concepts of monochromatic connection number $mc(G)$ and monochromatic vertex connection number $mvc(G)$ are natural opposite concepts of rainbow connection number $rc(G)$ and rainbow vertex connection number $rvc(G)$. For details about them we refer to a book [10] and a survey paper [9]. Moreover, the concept of total rainbow connection number $trc(G)$ in [12] was motivated by the rainbow connection number $rc(G)$ and rainbow vertex connection number $rvc(G)$. Thus, here we introduce the concept of total monochromatic connection of graphs. A graph is said to be *total-colored* if all the edges and the vertices of the graph are colored. A path in a total-colored graph is a *total monochromatic path* if all the edges and internal vertices on the path have the

same color. A total-coloring of a graph is a *total monochromatically-connecting coloring* (TMC-coloring, for short) if any two vertices of the graph are connected by a total monochromatic path of the graph. For a connected graph G , the *total monochromatic connection number*, denoted by $tmc(G)$, is defined as the maximum number of colors used in a TMC-coloring of G . An *extremal TMC-coloring* is a TMC-coloring that uses $tmc(G)$ colors. It is easy to check that $tmc(G) = m + n$ if and only if G is a complete graph.

The rest of this paper is organized as follows: In Section 2, we prove that $tmc(G) \geq m - n + 2 + l(G)$ for any connected graph and determine the value of $tmc(G)$ for some special graphs. In Section 3, we prove that there are many graphs with $tmc(G) = m - n + 2 + l(G)$ which are restricted by other graph parameters such as the maximum degree, the diameter and so on, and moreover, we show that for almost all graphs G , $tmc(G) = m - n + 2 + l(G)$ holds. In Section 4, we compare $tmc(G)$ with $mvc(G)$ and $mc(G)$, respectively, and obtain that there exist graphs G such that $tmc(G)$ is not less than $mvc(G)$ and vice versa, and that $tmc(G) = mc(G) + l(G)$ for almost all graphs. Moreover, we prove that $tmc(G) \leq mc(G) + mvc(G)$, and the equality holds if and only if G is a complete graph.

2 Preliminary results

In this section, we show that $tmc(G) \geq m - n + 2 + l(G)$ and present some preliminary results on the total monochromatic connection number. Moreover, we determine the value of $tmc(G)$ when G is a tree, a wheel, and a complete multipartite graph. The following fact is easily seen.

Proposition 1. *If G is a connected graph and H is a connected spanning subgraph of G , then $tmc(G) \geq e(G) - e(H) + tmc(H)$.*

Since for any two vertices of a tree, there exists only one path connecting them, we have the following result.

Proposition 2. *If T is a tree, then $tmc(T) = l(T) + 1$.*

The consequence below is immediate from Propositions 1 and 2.

Theorem 1. *For a connected graph G , $tmc(G) \geq m - n + 2 + l(G)$.*

Let G be a connected graph and f be an extremal TMC-coloring of G that uses a given color c . Note that the subgraph H formed by the edges and vertices colored c is connected, or we will give a fresh color to all the edges and vertices colored c in some of these components while still maintaining a TMC-coloring. Moreover, the color of each

internal vertex of H is c . Otherwise, let u_1, \dots, u_t be the internal vertices of H such that each of them is not colored c . We obtain the subgraph H_0 by deleting the vertices $\{u_1, \dots, u_t\}$. If H_0 is connected, it is possible to choose an edge incident with u_1 and assign it with a fresh color while still maintaining a TMC-coloring. If not, we can give a fresh color to all the edges and vertices colored c in some of these components while still maintaining a TMC-coloring. Furthermore, H does not contain any cycle; otherwise, a fresh color can be assigned to any edge of the cycle while still maintaining a TMC-coloring. Thus, H is a tree where the color of each internal vertex is c . Now we define the *color tree* as the tree formed by the edges and vertices colored c , denoted by T_c . If T_c has at least two edges, the color c is called *nontrivial*. Otherwise, c is *trivial*. We call an extremal TMC-coloring *simple* if for any two nontrivial colors c and d , the corresponding trees T_c and T_d intersect in at most one vertex. If f is simple, then the leaves of T_c must have distinct colors different from color c . Otherwise, we can give a fresh color to such a leaf while still maintaining a TMC-coloring. Moreover, a nontrivial color tree of f with m' edges and q' internal vertices is said to *waste* $m' - 1 + q'$ colors. For the rest of this paper we will use these facts without further mentioning them.

The lemma below shows that one can always find a simple extremal TMC-coloring for a connected graph.

Lemma 1. *Every connected graph G has a simple extremal TMC-coloring.*

Proof. We are given an extremal TMC-coloring f of G with the most number of trivial colors, and then we prove that this coloring must be simple. Suppose that there exist two nontrivial colors c and d such that T_c and T_d contain k common vertices denoted by u_1, u_2, \dots, u_k , where $k \geq 2$. Now we divide our discussion into two cases.

Case 1. For $1 \leq i \leq k$, u_i is an internal vertex of T_c or T_d .

For $1 \leq i \leq k$, if u_i is an internal vertex of T_c , u_i must be a leaf of T_d and then set $e_i = u_i w_i$ where w_i is the neighbor of u_i in T_d ; otherwise, u_i must be a leaf of T_c and then put $e_i = u_i v_i$ where v_i is the neighbor of u_i in T_c . Let H denote the subgraph consisting of the edges and vertices of $T_c \cup T_d$. Clearly, H is connected. We obtain a spanning tree H_0 of H by deleting the edges $\{e_2, e_3, \dots, e_k\}$. Now we change the total-coloring of H while still maintaining the colors of the leaves in H_0 unchanged. Assign the edges and internal vertices of H_0 with color c and the remaining edges $\{e_2, e_3, \dots, e_k\}$ with distinct new colors. Obviously, the new total-coloring is also a TMC-coloring and uses $k - 2$ more colors than our original one. So, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on f .

Case 2. There exists a vertex among u_1, \dots, u_k , say u_1 , which is a leaf of both T_c and T_d .

Let v_1 and w_1 be the neighbors of u_1 in T_c and T_d , respectively. There must be another color tree T_e (including a single edge) connecting v_1 and w_1 . For $1 \leq i \leq k$, if u_i is a leaf of T_c , then set $e_i = u_i v_i$ where v_i is the neighbor of u_i in T_c ; otherwise, u_i must be a leaf of T_d and then put $e_i = u_i w_i$ where w_i is the neighbor of u_i in T_d . Let H_1 denote the subgraph consisting of the edges and vertices of $T_c \cup T_d$. We obtain a spanning subgraph H_2 of H_1 by deleting the edges $\{e_1, e_2, \dots, e_k\}$. If T_e and H_2 do not have common leaves, let $E_0 = \{e_1, e_2, \dots, e_k\}$. Otherwise, let u'_1, \dots, u'_t denote the common leaves of T_e and H_2 . Set $e'_i = u'_i v'_i$ where v'_i is the neighbor of u'_i in T_e for $1 \leq i \leq t$. And then let $E_0 = \{e_1, \dots, e_k, e'_1, \dots, e'_t\}$. Let H denote the subgraph consisting of the edges and vertices of $T_c \cup T_d \cup T_e$. Clearly, H is connected. We obtain a spanning connected subgraph H_0 of H by deleting the edges of E_0 . Now we change the total-coloring of H while still maintaining the colors of the leaves in H_0 unchanged. Assign the edges and internal vertices of H_0 with color c and the remaining edges of H (i.e., the edges of E_0) with distinct new colors. Note that if v is a common leaf of either T_c and T_d or T_e and H_2 , it is also a leaf of H_0 . Obviously, the new total-coloring is also a TMC-coloring and uses at least $k + t - 2$ more colors than our original one. So, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on f . \square

Now we use the above results to compute the total monochromatic connection numbers of wheel graphs and complete multipartite graphs.

Example 1. Let G be a wheel W_{n-1} of order $n \geq 5$. Then $tmc(G) = m - n + 2 + l(G)$.

Proof. We are given a simple extremal TMC-coloring f of G . Note that $m - n + 2 + l(G) = m + 1$ and $tmc(G) \geq m + 1$ by Theorem 1. Suppose that f consists of k nontrivial color trees, denoted by T_1, \dots, T_k . In fact, we can always find two vertices with degree at least 4 if $k \geq 3$, a contradiction. Likewise, if $k = 2$, G must be W_4 and $tmc(W_4) = m + 1$. Thus, assume that $k = 1$ and T_1 is not spanning (Otherwise, $tmc(G) = m - n + 2 + l(G)$). Note that for every vertex $v \notin T_1$, there exist the total monochromatic paths connecting v and the $|T_1|$ vertices of T_1 . As f is simple, these paths are internally vertex-disjoint. Hence, $\deg(v) \geq |T_1|$. If $|T_1| \geq 4$, the $n - 1$ vertices with degree 3 of G must be in T_1 and then T_1 is a path. Thus, $tmc(G) = m + n - (n - 3) - (n - 3) = m + 6 - n \leq m + 1$. If $|T_1| = 3$, then G must be W_3 while $n \geq 5$. Therefore, the proof is complete. \square

Example 2. Let $G = K_{n_1, \dots, n_r}$ be a complete multipartite graph with $n_1 \geq \dots \geq n_t \geq 2$ and $n_{t+1} = \dots = n_r = 1$. Then $tmc(G) = m + r - t$.

Proof. The case that $r = 2$ is a special case of Theorem 3 whose proof is given in Section 3, so assume that $r \geq 3$. Let f be a simple extremal TMC-coloring of G with maximum

trivial colors. Suppose that f consists of k nontrivial color trees, denoted by T_1, \dots, T_k , where $t_i = |V(T_i)|$ and $q_i = q(T_i)$ for $1 \leq i \leq k$. Now we divide our discussion into two cases.

Case 1. $t = r$.

In this case, every vertex appears in at least one of the nontrivial color trees. Note that $m - n + 2 + l(G) = m$ and $tmc(G) \geq m$ by Theorem 1. If $\sum_{i=1}^k (t_i - 1) \geq n$, then we have that $tmc(G) \leq m + n - n - \sum_{i=1}^k q_i + k = m - \sum_{i=1}^k q_i + k \leq m$. Thus, $tmc(G) = m$. Suppose that $\sum_{i=1}^k (t_i - 1) \leq n - 1$. Now consider the subgraph G' consisting of the union of the T_i and let C_1, \dots, C_s denote its components.

Now we may assume that there exists a component, say C_1 , such that each nontrivial color tree in C_1 is a star. Let S be a star of C_1 with center u and leaves u_1, \dots, u_p , where $u_1, \dots, u_{p'}$ are in the same vertex class, say V_1 . Suppose that $p' \geq 2$. Indeed, if $p' = 1$, we can give a new color to the edge uu_1 while still maintaining a TMC-coloring. We claim that C_1 contains a cycle. If $p' < |V_1|$, there exists a vertex u_{p+1} of V_1 not adjacent to u in S . Then u_1 and u_{p+1} must be in a same nontrivial color tree and the same happens for $u_{p'}$ and u_{p+1} . These nontrivial color trees containing u_1 , $u_{p'}$ and u_{p+1} must form a cycle. If $p' = |V_1|$, we have that the vertices of the vertex class containing u must be in a same nontrivial color tree, or we will get a cycle in a similar way. By that analogy, we obtain a cycle formed by some centers of the nontrivial color trees in C_1 . Now we change the total-coloring of C_1 . We obtain a spanning tree T' of C_1 by connecting u_1 to the vertices in the same class with u and u to the other vertices of C_1 . We color the edges and internal vertices of T' with the same color and all other edges and vertices with distinct new colors. Clearly, this new total-coloring is also a TMC-coloring. However, it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on f .

Thus, suppose that there exists a nontrivial color tree of C_i , say T_{i1} , having two adjacent internal vertices u_i and v_i for $1 \leq i \leq s$. We obtain a spanning tree T by connecting v_1 to each vertex in the same class with u_1 of G and u_1 to the other vertices of G . Now we give a new total-coloring f' of G . Color the edges and internal vertices of T with the same color and all other edges and vertices of G with distinct new colors. Obviously, f' is still a TMC-coloring. If $s \geq 2$, then it either uses more colors or uses the same number of colors but more trivial colors than f , a contradiction. Thus, $s = 1$. Moreover, we can check that f' is a simple extremal TMC-coloring with maximum trivial colors. Therefore, $tmc(G) = m$.

Case 2. $t < r$.

We obtain a star S by connecting a vertex of $\cup_{i=t+1}^r V_i$ to each vertex of $\cup_{i=1}^t V_i$. Color

the edges and the center vertex of S with the same color and all other edges and vertices of G with distinct new colors. Clearly, this new total-coloring is still a TMC-coloring, denoted by f' . Thus, $tmc(G) \geq m + r - t$. If $\sum_{i=1}^k (t_i - 1) \geq n - r + t$, then we have that $tmc(G) \leq m + n - (n - r + t) - \sum_{i=1}^k q_i + k = m + r - t - \sum_{i=1}^k q_i + k \leq m + r - t$. Hence, $tmc(G) = m + r - t$. Suppose that $\sum_{i=1}^k (t_i - 1) \leq n - r + t - 1$. Next consider the subgraph G' consisting of the union of the T_i 's and suppose that it has s components, say C_1, \dots, C_s . Note that $|V(G')| \geq n - r + t$ since any two vertices of the same class must be covered in a nontrivial color tree. The case that $|V(G')| = n - r + t$ can be verified by a similar discussion to Case 1. Thus, suppose that $|V(G')| > n - r + t$. It is obvious that $s \geq 2$. Moreover, there must exist a vertex x of $\cup_{i=t+1}^r V_i$, which is contained in a component of G' , say C_1 . For $2 \leq j \leq s$, there does not exist a vertex of $\cup_{i=t+1}^r V_i$ in C_j . Otherwise, let x be the center of S and then f' either uses more colors or uses the same number of colors but more trivial colors than f , a contradiction. By a similar discussion to Case 1, we can obtain that there exists a nontrivial color tree of C_j having two adjacent internal vertices for $2 \leq j \leq s$. We obtain a star S_1 by joining the vertices of $\cup_{i=2}^s C_i$ to one internal vertex of C_1 . We give a new total-coloring of G while still maintaining the total-coloring of C_1 unchanged. Assign the edges and the center vertex of S_1 with one color and the other edges and vertices of $G \setminus C_1$ with distinct new colors. This new total-coloring is still a TMC-coloring and it either uses more colors or uses the same number of colors but more trivial colors, contradicting the assumption on f . Therefore, we have finished the proof. \square

3 Graphs with $tmc(G) = m - n + 2 + l(G)$

In this section, we prove that there are many graphs G for which $tmc(G) = m - n + 2 + l(G)$, even for almost all graphs.

Lemma 2. [6] *Let G be a connected graph of order $n > 3$. If G satisfies any of the following properties, then $mc(G) = m - n + 2$.*

- (a) *The complement \overline{G} of G is 4-connected.*
- (b) *G is K_3 -free.*
- (c) *$\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$. In particular, this holds if $\Delta(G) \leq (n+1)/2$, and this also holds if $\Delta(G) \leq n - 2m/n$.*
- (d) *$\text{diam}(G) \geq 3$.*
- (e) *G has a cut vertex.*

We can obtain that $tmc(G) \leq mc(G) + l(G)$ for a noncomplete graph, whose proof is

contained in the proof of Theorem 6 in Section 4. In addition with Theorem 1 and Lemma 2, we have the following results.

Theorem 2. *Let G be a connected graph of order $n > 3$. If G satisfies any of the following properties, then $tmc(G) = m - n + 2 + l(G)$.*

- (a) *The complement \overline{G} of G is 4-connected.*
- (b) *G is K_3 -free.*
- (c) *$\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$.*
- (d) *$\text{diam}(G) \geq 3$.*
- (e) *G has a cut vertex.*

One cannot hope to strengthen Theorem 2(c) by improving the upper bound of $\Delta(G)$. In fact, let $G = K_{n-2,1,1}$. Then we have that $tmc(G) = m - n + 3 + l(G)$ while the maximum degree is $n - 1 = n - \frac{2m-3(n-1)}{n-3}$.

From Theorem 2(a), we can get a stronger result. For a property P of graphs and a positive integer n , define $\text{Prob}(P, n)$ to be the ratio of the number of graphs with n labeled vertices having P over the total number of graphs with these vertices. If $\text{Prob}(P, n)$ approaches 1 as n tends to infinity, then we say that *almost all* graphs have the property P . See [1] for example.

Theorem 3. *For almost all graphs G , we have that $tmc(G) = m - n + 2 + l(G)$.*

In order to prove Theorem 3, we need the following lemma.

Lemma 3. [1] *For every nonnegative integer k , almost all graphs are k -connected.*

Proof of Theorem 3: For any given nonnegative integer n , let \mathcal{G}_n denote the set of all graphs of order n , and let \mathcal{G}_n^4 denote the set of all 4-connected graphs of order n . Moreover, let \mathcal{B}_n denote the set of all graphs G of order n such that the complement \overline{G} of G is 4-connected. Note that for any two graphs G and H , $G \cong H$ if and only if $\overline{G} \cong \overline{H}$. Then, it is easy to check that the map: $G \rightarrow \overline{G}$ is a bijection from \mathcal{B}_n to \mathcal{G}_n^4 . Therefore, we have

$$\frac{|\mathcal{B}_n|}{|\mathcal{G}_n|} = \frac{|\mathcal{G}_n^4|}{|\mathcal{G}_n|}.$$

By Lemma 3, it follows that almost all graphs are 4-connected. Then, we get that almost all graphs have 4-connected complements. Furthermore, since almost all graphs are connected, we have that $tmc(G) = m - n + 2 + l(G)$ by Theorem 2(a). \square

Remark 1. For the monochromatic connection number $mc(G)$, from Lemma 2(a) and Lemma 3, one can deduce, in a similar way, that for almost all graphs G , $mc(G) = m - n + 2$ holds.

Remark 2. To use the parameter $l(G)$ in the above formulas looks good. However, from [7, p.206] we know that it is NP-hard to find a spanning tree that has maximum number of leaves in a connected graph G .

4 Compare $tmc(G)$ with $mvc(G)$ and $mc(G)$

Let G be a nontrivial connected graph. Firstly, we compare $tmc(G)$ with $mvc(G)$. The question we may ask is, can we bound one of $tmc(G)$ and $mvc(G)$ in terms of the other? The following two theorems give sufficient conditions for $tmc(G) > mvc(G)$.

Theorem 4. *Let G be a connected graph with diameter d . If $m \geq 2n - d - 2$, then $tmc(G) > mvc(G)$.*

Proof. The case that $d = 1$ is trivial, so assume that $d \geq 2$. We can check that if $l(G) = 2$, then $tmc(G) > mvc(G)$. Thus, suppose that $l(G) \geq 3$. By Theorem 1, it follows that $tmc(G) \geq m - n + 2 + l(G) \geq 2n - d - 2 - n + 2 + 3 = n - d + 3$. Moreover, we have that $mvc(G) \leq n - d + 2$ by [5, Proposition 2.3]. Therefore, $tmc(G) > mvc(G)$. \square

Theorem 5. *Let G be a connected graph of diameter 2 with maximum degree Δ . If $\Delta \geq \frac{n+1}{2}$, then $tmc(G) > mvc(G)$.*

Before proving Theorem 5, we need the lemma below.

Lemma 4. [2] *Let G be a connected graph of diameter 2 with maximum degree Δ . Then*

$$m \geq \begin{cases} n + \Delta - 2, & \text{if } \Delta = n - 2 \text{ or } n - 3 \\ 2n - 5, & \text{if } \Delta = n - 4 \\ 2n - 4, & \text{if } \frac{2n-2}{3} \leq \Delta \leq n - 5 \\ 3n - \Delta - 6, & \text{if } \frac{3n-3}{5} \leq \Delta < \frac{2n-2}{3} \\ 5n - 4\Delta - 10, & \text{if } \frac{5n-3}{9} \leq \Delta < \frac{3n-3}{5} \\ 4n - 2\Delta - 11, & \text{if } \frac{n+1}{2} \leq \Delta < \frac{5n-3}{9} \end{cases} \quad (1)$$

Proof of Theorem 5: The case that $n \leq 7$ can be easily verified. Suppose that $n \geq 8$. Since the diameter of G is 2, we have that $mvc(G) = n$. By Theorem 1 and Lemma 4, $tmc(G) \geq m - n + 2 + l(G) > n$. Thus, $tmc(G) > mvc(G)$. \square

Actually, we have that $tmc(C_5) = 4 < mvc(C_5) = 5$, where $m < 2n - d - 2$ and $\Delta < \frac{n+1}{2}$. This implies that the conditions of Theorems 4 and 5 cannot be improved. Moreover, if G is a star, then $tmc(G) = mvc(G) = n$. Therefore, there exist graphs G such that $tmc(G)$ is not less than $mvc(G)$ and vice versa. However, we cannot show whether there exist other graphs with $tmc(G) \leq mvc(G)$. Thus, we propose the following problem.

Problem 1. *Dose there exist a graph of order $n \geq 6$ except a star such that $tmc(G) \leq mvc(G)$?*

Next we compare $tmc(G)$ with $mc(G)$. If G satisfies one of the conditions in Theorem 2, then we have $mc(G) = m - n + 2$ and so $tmc(G) = mc(G) + l(G)$. For a complete graph G , $tmc(G) > mc(G) + l(G)$. From [6, Corollary 13], if G is a wheel W_{n-1} of order $n \geq 5$, we have that $mc(G) = m - n + 3$ and then $tmc(G) < mc(G) + l(G)$. However, by Theorem 3 and Remark 1, it follows that almost all graphs have that $tmc(G) = mc(G) + l(G)$ which implies that almost all graphs have that $tmc(G) > mc(G)$. Thus, we propose the following conjecture.

Conjecture 1. For a connected graph G , it always holds that $tmc(G) > mc(G)$.

Finally, we compare $tmc(G)$ with $mc(G) + mvc(G)$.

Theorem 6. *Let G be a connected graph. Then $tmc(G) \leq mc(G) + mvc(G)$, and the equality holds if and only if G is a complete graph.*

In order to prove Theorem 6, we need the following lemma.

Lemma 5. *For a noncomplete connected graph G , let f be a simple extremal TMC-coloring of G and T_1, \dots, T_k denote all the nontrivial color trees of f , where $t_i = |V(T_i)|$ and $q_i = q(T_i)$ for $1 \leq i \leq k$. Then, $\sum_{i=1}^k q_i \geq q(G)$.*

Proof. For any $v \in G$, if $v \notin \cup_{i=1}^k T_i$, v must be adjacent to an internal vertex w_0 of a nontrivial color tree and then set $E_v = \{vw|w \in N(v) \setminus \{w_0\}\}$. If v is an internal vertex of a nontrivial color tree containing v , set $E_v = \emptyset$. Otherwise, v is a leaf of any nontrivial color tree containing v . Let T_1, \dots, T_s denote the nontrivial color trees containing v and v_1, \dots, v_s be the neighbors of v in T_1, \dots, T_s , respectively. Let $E_v = \{vv_1, \dots, vv_s\}$. We obtain a spanning subgraph G' by deleting the edges of $\bigcup_{v \in G} E_v$. Note that every vertex of $\{v : E_v = \emptyset\}$ is connected to each other. For any two vertices u_1 and u_2 of $\{v : E_v = \emptyset\}$, there exists a total monochromatic path P of G connecting them. For each vertex u of P , we have $E_u = \emptyset$. Thus, G' also contains P from u_1 to u_2 . Moreover, every vertex of $\{v : E_v \neq \emptyset\}$ is connected to a vertex of $\{v : E_v = \emptyset\}$. Hence, G' is connected and each vertex of $\{v : E_v \neq \emptyset\}$ cannot be an internal vertex of G' . Then $\sum_{i=1}^k q_i \geq q(G') \geq q(G)$. \square

Now, we are ready to prove Theorem 6.

Proof of Theorem 6: If G is a complete graph, we have that $tmc(G) = mc(G) + mvc(G)$. Thus, suppose that G is not complete. We are given a simple extremal TMC-coloring f of G . Suppose that f consists of k nontrivial color trees denoted by T_1, \dots, T_k , where

$t_i = |V(T_i)|$ and $q_i = q(T_i)$ for $1 \leq i \leq k$. Then $tmc(G) = m + n - \sum_{i=1}^k (t_i - 2) - \sum_{i=1}^k q_i$. Now we take a copy G' of G . Then G' contains the trees T'_1, \dots, T'_k corresponding to T_1, \dots, T_k , respectively. Define an edge-coloring f_e of G' as follows: color the edges of T_i with color i for $1 \leq i \leq k$ and the other edges of G' with distinct new colors. Then f_e is an MC-coloring of G' with $m - \sum_{i=1}^k (t_i - 2)$ colors. Thus, $mc(G) = mc(G') \geq m - \sum_{i=1}^k (t_i - 2) = tmc(G) - n + \sum_{i=1}^k q_i$. By Lemma 5, we have that $\sum_{i=1}^k q_i \geq q(G)$. Then $tmc(G) \leq mc(G) + n - q(G) = mc(G) + l(G)$. Moreover, it is easy to obtain that $mvc(G) \geq l(G) + 1$. Hence, $tmc(G) < mc(G) + mvc(G)$. Therefore, the proof is complete. \square

Remark 3. For the total rainbow connection number $trc(G)$, we cannot bound one of $trc(G)$ and $rc(G) + rvc(G)$ in terms of the other. For a connected graph G , $trc(G) = rc(G) + rvc(G)$ if G is a complete graph or a star. Moreover, if G is a complete bipartite graph $K_{m,n}$ with $m \geq 2$ and $n \geq 6^m$, then $trc(G) = 7 > rc(G) + rvc(G) = 4 + 1$ [9, 11, 12]. In [12], for every $s \geq 1481$, there exists a graph G with $trc(G) = rvc(G) = s$ which implies that $trc(G) < rc(G) + rvc(G)$. This is one thing that the total monochromatic connection differs from the total rainbow connection.

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